

MIT-CTP-3140

Photon Decay at the Schwarzschild Horizon

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Abstract

A recent proposal that gravity theory is an emergent phenomenon also entails the possibility of photon decay near the Schwarzschild event horizon. We present a possible mechanism for such decay, which utilizes a dimensional reduction near the horizon.

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In a recent paper [1], the question is raised whether classical General Relativity can exist as an emergent phenomenon—as the low-energy limit of an underlying quantum system. In this view, the singularity at the Schwarzschild event horizon represents a failure of the effective description owing to the divergence of a characteristic coherence length. We shall introduce an additional element to this model: a natural change in the dimensionality of virtual particle loop integrals of the quantum system near the horizon. This change will have important implications. In particular, it may cause photons to decay when they near the event horizon of a black hole, as suggested in [1].

In the Schwarzschild metric, the line element is

$$ds^2 = \left(1 - \frac{r_S}{r}\right) dt^2 - \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2). \quad (1)$$

(The velocity of light is set to one.) Classically, this metric concentrates the motion in the radial direction near $r = r_S$. This can be seen by examining the spatial line element

$$dl^2 = \left(1 - \frac{r_S}{r}\right)^{-1} \left[dr^2 + \left(1 - \frac{r_S}{r}\right) r^2 d\Omega^2 \right]. \quad (2)$$

For r near r_S , the angular variables are suppressed, and motion is confined to the two-dimensional t - r subspace.

More specifically, consider the geodesic equation, $\ddot{x}^\mu + \Gamma_{\nu\rho}^\mu \dot{x}^\nu \dot{x}^\rho = 0$, where \dot{x}^μ is the derivative of the position with respect to an affine parameter.

The temporal and angular equations,

$$\ddot{t} + \frac{r_S}{r(r - r_S)} \dot{t}\dot{r} = 0, \quad (3)$$

$$\ddot{\theta} + 2\frac{\dot{r}}{r}\dot{\theta} - \frac{1}{2}\sin 2\theta \dot{\phi}^2 = 0, \quad (4)$$

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot \theta \dot{\theta}\dot{\phi} = 0, \quad (5)$$

can be readily integrated to give

$$\dot{t} \frac{r - r_S}{r} = \tau, \quad (6)$$

$$r^2 \sin^2 \theta \dot{\phi} = m, \quad (7)$$

$$(r^2 \dot{\theta})^2 + \frac{m^2}{\sin^2 \theta} = l^2. \quad (8)$$

τ , m , and l^2 are integration constants. Near the Schwarzschild horizon we may allow all particles to be massless and take the geodesics to be null. Then the constants will diverge, but we retain their ratios, which remain finite. From the θ equation, it is clear that for $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$, $\ddot{\theta} = 0$ also, so the motion remains in the equatorial plane, with $m^2 = l^2$.

Using the other three equations, the radial equation

$$\ddot{r} + \left(1 - \frac{r_S}{r}\right) \frac{r_S}{2r^2} \dot{t}^2 + \left(1 - \frac{r_S}{r}\right)^{-1} \frac{r_S}{2r^2} \dot{r}^2 + (r_S - r)(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) = 0, \quad (9)$$

becomes

$$\ddot{r} + \frac{\tau^2 r_S}{2r(r - r_S)} - \frac{r_S}{2r(r - r_S)} \dot{r}^2 - \frac{l^2(r - r_S)}{r^4} = 0. \quad (10)$$

In this form, the equation may be integrated to give

$$\dot{r} = \pm \tau \sqrt{1 - \left(1 - \frac{r_S}{r}\right) \frac{\lambda^2}{r^2}}, \quad (11)$$

where $\lambda = l/\tau$ and a possible integration constant is set to zero for null geodesics.

Dividing the ϕ and r equations, (7) and (11), by the t equation (6), gives at $\theta = \frac{\pi}{2}$, $m = l$,

$$\frac{d\phi}{dt} = \left(1 - \frac{r_S}{r}\right) \frac{\lambda}{r^2}, \quad (12)$$

$$\frac{dr}{dt} = \pm \left(1 - \frac{r_S}{r}\right) \sqrt{1 - \left(1 - \frac{r_S}{r}\right) \frac{\lambda^2}{r^2}}. \quad (13)$$

Consider a photon trajectory just outside the event horizon, at radius $r = r_S + \Delta$. For $\Delta \ll r_S$, the time-evolution equations (12) and (13) become

$$\frac{d\Delta}{dt} = \pm \frac{\Delta}{r_S} \sqrt{1 - \frac{\lambda^2}{r_S^3} \Delta}, \quad (14)$$

$$\frac{d\phi}{dt} = \frac{\lambda}{r_S^3} \Delta. \quad (15)$$

The equations, with the lower sign in (14), may be integrated, yielding

$$\Delta = \frac{r_S^3 / \lambda^2}{\cosh^2 \frac{t}{2r_S}} \quad (16)$$

$$\phi = \frac{2r_S}{\lambda} \tanh \frac{t}{2r_S}. \quad (17)$$

Equations (16) and (17) show that if $\frac{\lambda}{r_S}$ is sufficiently small ($\ll \pi$), then the radial motion will be much more rapid than the angular motion. This is the regime we are interested in, where the concentration of motion in the radial direction reduces some aspects of the system to 1+1 dimensions.

We shall consider a much more profound reduction in the dimensionality of the system. Taking local coordinates r , x , and y , the spatial line element is

$$dl^2 = -g_{rr} dr^2 + dx^2 + dy^2. \quad (18)$$

In the model [1], the singularity at $r = r_S$ represents a real physical effect, not merely a coordinate artifact, so these are very natural linear coordinates. If we suppose that p^r , p^x , and p^y are cut off at the same scale in a loop integral, $\mathbf{p}^2 \equiv -p_i p^i$ will be dominated by $-g_{rr}(p^r)^2$. (This supposition strongly breaks general covariance, of course.) This is the kind of situation we would like to analyze. However, a sharp momentum cutoff is not gauge invariant; to study the photon self-energy, we shall translate this idea into gauge-invariant language, using dimensional regularization.

In the renormalization prescription described above, one of the spatial dimensions provides the dominant contribution to \mathbf{p}^2 . In the language of dimensional regularization, this can be seen as a reduction in the effective dimensionality d of the momentum integral to $d < 4$.

We shall find the effective dimensionality by examining the volume element of this system, because the momentum cutoff in a given direction and the volume contribution of that direction are closely related. To see this, consider for the moment a theory that is regulated by a lattice at short distances. The volume of a region counts the number of lattice points in that region. The lattice spacing a_i in a given direction governs the density of lattice points along that axis, so a length L in the x_i -direction contributes an amount $\alpha \frac{L}{a_i}$ to the volume, where α is a scaling constant independent of direction. The lattice spacing also corresponds directly to the momentum cutoff in that direction, $p_i^{max} = \frac{\pi}{a_i}$. So the dependence of the volume on a given dimension and the momentum cutoff in that direction are intimately linked.

So we look for an expression for the effective dimensionality (to be used in dimensional regularization) in terms of the volume element. In Schwarzschild space-time, the spatial volume element is

$$dV = \sqrt{g} dr d\theta d\phi. \quad (19)$$

Here, $g = -g_{rr}r^4 \sin^2 \theta$ is the determinant of the spatial metric $-g_{ij}$. The classical geodesic problem suggests that the radial direction should always contribute one effective dimension, while the angular directions may contribute less than one. We wish to determine the effective dimensionality by an integral over an effective volume. To achieve a reasonable result, the effective volume is defined by rescaling a factor of $\sqrt{-g_{rr}}$ from each direction. So the new volume element reads

$$dV' = \frac{r^2 \sin \theta}{-g_{rr}} dr d\theta d\phi = g_{tt} dV_E, \quad (20)$$

where dV_E is a Euclidean volume element.

From dV' , we need to find an expression for the number of effective dimensions. This expression should have several properties. The dimension corresponding to a volume element $dx_1 dx_2 \cdots dx_n$ should be n . (with dV_E corresponding to 3 dimensions). So the dimension function should be additive where the volume element is multiplicative; this is the fundamental property of a logarithm. So a natural choice for the effective spatial

dimension d_s is

$$d_s = \frac{\ln \int_0^\Lambda dV'}{\ln \Lambda}. \quad (21)$$

Evaluating this gives us

$$d_s = \frac{\ln \int_0^\Lambda dV_E}{\ln \Lambda} + 2 \frac{\ln \sqrt{g_{tt}}}{\ln \Lambda} = 3 + 2 \frac{\ln \sqrt{g_{tt}}}{\ln \Lambda}. \quad (22)$$

Equation (22) has some problems. The most striking one is that Λ appears to be a dimensional quantity, which would make $\frac{\ln \sqrt{g_{tt}}}{\ln \Lambda}$ ambiguous. The obvious solution is that the coordinates must be nondimensionalized, to make Λ dimensionless. However, it is not at all obvious how to nondimensionalize the coordinates. Fortunately, we do not need to deal with that question directly. Regardless of the coordinates' dimensions, dV_E will always contribute three dimensions to d_s . To analyze the last term in (22), we introduce the natural condition that the angular factors in the volume element can not contribute any less than zero dimensions each. That is,

$$\frac{\ln \sqrt{g_{tt}}}{\ln \Lambda} \geq -1. \quad (23)$$

Since $g_{tt} < 1$ and the logarithm is strictly increasing, this condition may be rewritten as

$$\Lambda^{-1} \leq \sqrt{g_{tt}}. \quad (24)$$

In classical General Relativity, $g_{tt}(r_S) = 0$, but in the model [1], g_{tt} drops to a nonzero minimum value

$$g_{tt}^{min} \sim 1 - \frac{r_S}{r_S + \delta} \approx \frac{\delta}{r_S}, \quad (25)$$

where δ is a small length that characterizes how the classical Schwarzschild singularity is cut off by the underlying quantum system. We expect δ to be related to the Planck length. Thus we estimate $\Lambda \sim \sqrt{r_S/\delta}$. Close to the event horizon, at $r = r_S + \Delta$, where $\delta \ll \Delta \ll r_S$ and $g_{tt} \sim \Delta/r_S$, the effective dimension of the system is

$$d_s = 1 + 2 \left(1 - \frac{\ln \Delta/r_S}{\ln \delta/r_S} \right) = 1 + 2 \left(\frac{\ln \delta/\Delta}{\ln \delta/r_S} \right). \quad (26)$$

So we will look at the problem of Quantum Electrodynamics in $(1 + \epsilon) + 1$ dimensions, where

$$\epsilon = 2 \left(\frac{\ln \delta/\Delta}{\ln \delta/r_S} \right). \quad (27)$$

We shall only consider the contribution to the photon self-energy from massless particles. This should be a good approximation near the event horizon. Photons coming in from spatial infinity are highly blueshifted at $r = r_S + \Delta$, so the momentum scale in

the photon propagator is large compared to any invariant momentum scale (such as the electron mass). The same argument may also be phrased in different terms. The energy of a comoving electron of mass m_e is $\sqrt{g_{tt}} m_e$, so near the event horizon, the apparent electron mass becomes small. So it is reasonable to consider massless particles.

The one-loop photon self-energy due to a single species of charged massless fermions is

$$i\Pi^{\mu\nu}(q) = 2 \text{tr I} (q^2 \eta^{\mu\nu} - q^\mu q^\nu) \int_0^1 dx \int_k \frac{x(1-x)}{[k^2 + q^2 x(1-x)]^2}. \quad (28)$$

Here, I is the unit matrix in spinor space, and the Minkowski metric is denoted by $\eta^{\mu\nu}$, to avoid confusion with the GR metric $g^{\mu\nu}$. Since $d \rightarrow 4$ as $r \rightarrow \infty$, the Dirac matrices should be four-dimensional. Equation (28), derived in the appendix, gives, upon a d -dimensional k -integration,

$$i\Pi^{\mu\nu}(q) = -i \text{tr I} (q^2 \eta^{\mu\nu} - q^\mu q^\nu) \frac{e^2}{(-q^2)^{2-d/2}} \frac{1}{(4\sqrt{\pi})^{d-1}} \frac{(1-d/2)\pi}{\sin \frac{\pi d}{2}} \frac{1}{\Gamma(\frac{d}{2} + \frac{1}{2})}. \quad (29)$$

There is a subtlety in the use of (29). In dimensional regularization, it is usual to reduce all aspects of the problem to d dimensions. In our case, only the loop integral is d -dimensional. There are still four Dirac matrices, and the photon remains a four-component field. However, so long as the external photon momentum q lies in the d -dimensional subspace, equation (29) remains valid, and the metric $\eta^{\mu\nu}$ is d -dimensional.

The $d = 4$ and $d = 2$ cases of equation (29) are well understood. Since we have $d = 2 + \epsilon$, we expand around $d = 2$. Evaluating equation (29) with this value of d , we get

$$i\Pi^{\mu\nu}(q) = -i(q^2 \eta^{\mu\nu} - q^\mu q^\nu) \frac{2e^2}{\pi} \frac{1}{(-q^2)^{1-\epsilon/2}} \left\{ \frac{\pi^{1/2-\epsilon/2}}{2^{1+2\epsilon}} \frac{\epsilon\pi/2}{\sin \frac{\epsilon\pi}{2}} \frac{1}{\Gamma(\frac{3}{2} + \frac{\epsilon}{2})} \right\}. \quad (30)$$

The bracketed term in (30) is unity at $\epsilon = 0$. It is purely real, so will only contribute higher-order corrections to the real and imaginary parts of the self-energy.

At $d = 2$, we get Schwinger's well-known result that the photon becomes massive [2]. The self-energy is

$$i\Pi^{\mu\nu}(q)|_{d=2} = i \frac{2e^2}{\pi} \left(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right). \quad (31)$$

(This differs by a factor of two from the usual result, because here we have used four-dimensional Dirac matrices.) The residue of the pole at $q^2 = 0$ gives the photon mass $m_\gamma^2 = \frac{2e^2}{\pi}$.

For $d = 2 + \epsilon$, the result is only slightly different. Instead of having $\frac{1}{-q^2}$, we have $\frac{1}{(-q^2)^{1-\epsilon/2}}$, which we expand about $\epsilon = 0$ to get

$$\frac{1}{(-q^2)^{1-\epsilon/2}} \approx \frac{1}{-q^2} \left[1 + \frac{\epsilon}{2} \ln(-q^2) \right]. \quad (32)$$

As in equation (31), there is a pole at $q^2 = 0$. However, the residue is shifted by the second term in (32). To zeroth order in ϵ , the pole in the propagator is shifted to $\frac{2e^2}{\pi}$. To first order, the pole is further shifted to the value of $[1 + \frac{\epsilon}{2} \ln(-q^2)]$ evaluated at the new pole location. This shifts the pole to

$$\begin{aligned} m_\gamma^2 &= \frac{2e^2}{\pi} \left[1 + \frac{\epsilon}{2} \ln \left(-\frac{2e^2}{\pi} \right) \right] \\ &= \frac{2e^2}{\pi} \left[1 + \frac{\epsilon}{2} \ln(-1) + \frac{\epsilon}{2} \ln \left(\frac{2e^2}{\pi} \right) \right] \\ &\approx \frac{2e^2}{\pi} \left(1 - i \frac{\pi\epsilon}{2} \right). \end{aligned} \tag{33}$$

Equation (33) is correct to lowest order in ϵ in both its real and imaginary parts. The sign of the imaginary part has been chosen so that photons decay rather than appear.

We must now turn to the question of how to interpret equation (33). By expanding around $d = 2$, we have introduced a number of two-dimensional conventions. The e^2 appearing in (33) is the two-dimensional value of the electromagnetic coupling. In two dimensions, e has mass dimension one, so $\frac{2e^2}{\pi}$ is indeed a mass squared. We must relate the e in (33) (which we will henceforth refer to as e_2) to the four-dimensional electron charge e_4 .

We may relate the two charges by comparing the actions in two and four dimensions. In four dimensions, the electromagnetic Lagrangian density is

$$\mathcal{L}_4 = -\frac{1}{4e_4^2} F_{\mu\nu} F^{\mu\nu}. \tag{34}$$

Then the action is

$$S_4 = \int dt r^2 dr d\Omega \mathcal{L}_4. \tag{35}$$

The action S_2 derived from the two-dimensional Lagrangian \mathcal{L}_2 should be the same as S_4 , up to a constant factor C . So we have

$$\int dt dr \mathcal{L}_2 = -C \int dt r^2 dr d\Omega \frac{1}{4e_4^2} F_{\mu\nu} F^{\mu\nu}. \tag{36}$$

We must perform the angular integrals on the right-hand side of (36) to determine \mathcal{L}_2 . This means doing an integral over the submanifold orthogonal to the t - r subspace. This orthogonal submanifold is a sphere, and the integral over it will depend upon the radius at which the integral is evaluated. We are interested in radii $r \approx r_S$ (which is the only region where the integration over angles is justified). If $F^{\mu\nu}$ is spherically symmetric, the angular integral gives 4π , and we can set $r = r_S$ to get

$$\int dt dr \mathcal{L}_2 = -4\pi r_S^2 C \int dt dr \frac{1}{4e_4^2} F_{\mu\nu} F^{\mu\nu}. \tag{37}$$

We can now read off the value of \mathcal{L}_2 ,

$$\mathcal{L}_2 = -\frac{1}{4} \frac{4\pi r_S^2 C}{e_4^2} F_{\mu\nu} F^{\mu\nu}, \quad (38)$$

so that the two-dimensional charge is

$$e_2^2 = \frac{1}{4\pi r_S^2 C} e_4^2. \quad (39)$$

The constant C includes differences in how the field operators are normalized in two and four dimensions. So although the precise numerical relation between e_2 and e_4 has not been determined, the dependence of e_2 on r_S is unambiguous.

In deriving equation (39), we assumed that the field configuration was spherically symmetric. We can also evaluate the angular integral for more general field configurations, although this adds additional ambiguities. If the field $F^{\mu\nu}$ is in an $l > 0$, $m = 0$ multipole state, the integral becomes

$$\int dt dr \mathcal{L}_2 = -C \int dt r^2 dr d\Omega P_l(\cos\theta)^2 \frac{1}{4e_4^2} F_{\mu\nu} F^{\mu\nu}. \quad (40)$$

(The additional angular fields coming from derivatives of $P_l(\cos\theta)$ are suppressed by $\frac{1}{r_S}$ and have been dropped.) Since the maximum value of $P_l(\cos\theta)$ is $P_l(1) = 1$, the $F^{\mu\nu}$ appearing in equation (40) is the maximum value of the field over all angles. It is consistent with our earlier identification of the two- and four-dimensional fields to identify this $F^{\mu\nu}$ with the field appearing in \mathcal{L}_2 although other identifications could also be consistent). Evaluating the integral then gives us

$$e_2^2 = \frac{2l+1}{4\pi r_S^2 C} e_4^2. \quad (41)$$

A similar calculation may be done for $m \neq 0$, but the result (with these conventions) depends on m explicitly, not merely on l . Despite this problem, (41) remains a good candidate for an m -independent multipole field mass.

We must also address the question of how to interpret the imaginary part of (33). To help with the interpretation, we shall use an analogy to a much simpler dimensional reduction problem—the electromagnetic field in a rectangular waveguide [3]. This simple problem in classical electrodynamics has many similarities to the QED problem under consideration.

Consider a rectangular waveguide with metal walls. The waveguide has dimensions a in the x -direction and b in the y -direction. (We will presume that a and b are comparable in magnitude.) The waves propagate freely in the z -direction. The boundary conditions on this system restrict the wavevector of the electromagnetic field in the interior to be

$$\mathbf{k} = \frac{\pi n_x}{a} \hat{\mathbf{x}} + \frac{\pi n_y}{b} \hat{\mathbf{y}} + k_z \hat{\mathbf{z}}. \quad (42)$$

The numbers n_x and n_y are positive integers; at least one of n_x and n_y must be nonvanishing for fields to exist. The frequency $\omega = |\mathbf{k}|$ satisfies

$$\omega^2 = k_z^2 + \pi^2 \left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} \right). \quad (43)$$

Since propagation only occurs along the z -axis, it is natural to look at this system in the t - z subspace, where the wavevector is simply k_z . Then (43) looks like the energy-momentum relation for a relativistic particle of mass $m_{wg}^2 = \pi^2 \left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} \right)$.

So in $1 + 1$ dimensions, a photon in a waveguide acquires an effective mass. The scale of this mass is a^{-1} , where a is the characteristic size of the system in the neglected dimensions. This is the same scaling we found previously. In equation (33), the scale of the photon mass was r_S^{-1} , and r_S is the length scale of the event horizon in the angular directions. According to (41), the black hole system actually has a whole hierarchy of photon masses. The waveguide also exhibits this property; different n_x and n_y values give different values of m_{wg}^2 . (These results are similar to those found in Kaluza-Klein theories, although the higher modes are not strongly suppressed here.)

The waveguide system also exhibits another important property—decay. Through interactions in the x - and y -dimensions, a photon can disappear from the interior of the waveguide. This can occur in a variety of ways, depending on the regime. We mention the two simplest regimes and discuss the interpretation of decays in these regimes. At low frequencies, $\omega \ll \nu$, where ν is the collision frequency for electrons in the metal walls, the magnetic field drives surface currents which dissipate energy through resistive heating. This leads to a low-frequency energy loss

$$\Gamma \equiv \frac{\langle P \rangle}{\langle U \rangle} \propto \frac{1}{\sqrt{a^3 \sigma}}, \quad (44)$$

where σ is the conductivity of the metal walls. The behavior is different at higher frequencies, $\nu \ll \omega < \omega_p$, where ω_p is the plasma frequency, related to the electron density n_e by $\omega_p^2 = \frac{n_e e^2}{m_e}$. In this regime, the electromagnetic field is exponentially damped in the walls, but photons can tunnel through the walls and escape from the waveguide. However, the decay rate does not have a simple dependence on a and ω_p .

The decay rate in the waveguide depends strongly on the regime, but in each regime, the decay rate depends primarily on the length a and some other length parameter. In the regimes outlined above, the length parameters are provided by σ and ω_p . In the black hole model, the imaginary part of m_γ^2 is governed by ϵ , which depends on the inverse mass scale r_S , as well as on the lengths δ and Δ .

The two decay regimes outlined above possess very different decay processes. The first regime is dissipative. Through interactions, photons in the waveguide decay into something else—thermal excitations of the metal walls. In the second regime, the photon does not actually decay. Instead, it escapes from the waveguide and the corresponding $1 +$

1-dimensional subspace. Despite their differences, these processes would each contribute an imaginary part to m_{wg}^2 .

Analogously, the imaginary term in equation (33) could have two different origins. The photons could be decaying into other particles, as is suggested in [1]. Alternatively, the effective decay rate in (33) could correspond to photons being scattered out of the $1 + 1$ dimensional t - r subspace into states with large angular momenta. Our simple treatment does not allow us to distinguish between the two. However, either process would be novel—an effect caused by the finite minimum of g_{tt} and controlled in magnitude by δ .

Although it is not directly relevant to the problem at hand, it is worth mentioning one further suggestive aspect of the waveguide analogy. In $2 + 1$ dimensions, the photon self-energy (29) is still singular at $q^2 = 0$, but that singularity is weaker than a simple pole in q^2 [4]. This introduces ambiguity as to whether or not the photon has mass in $2 + 1$ dimensions. The waveguide analogy of $2 + 1$ -dimensional QED is the propagation of the electromagnetic field between two parallel plates of separation a in the x -direction. However, in this case, the wavevector in the x -direction, $\frac{\pi n_x}{a}$, is allowed to vanish. So a photon in this system may or may not behave as if it were massive.

Acknowledgments

This work is supported in part by funds provided by the U. S. Department of Energy (D.O.E.) under cooperative research agreement DE-FC02-94ER40818.

Appendix: Photon self-energy integral

To derive (28) in a dimension-independent manner, we begin with the integral

$$i\Pi^{\mu\nu}(q) = -e^2 \int_k \text{tr} \frac{\gamma^\mu (\not{k} + \not{q}) \gamma^\nu \not{k}}{(k+q)^2 k^2}. \quad (45)$$

Introducing a Feynman parameter x , setting $M^2 = -x(1-x)q^2$, shifting the integration variable $k \rightarrow k - xq$, and dropping all odd- k terms leaves

$$i\Pi^{\mu\nu} = -e^2 \int_0^1 dx \int_k \text{tr} \frac{\gamma^\mu \not{k} \gamma^\nu \not{k} - x(1-x) \gamma^\mu \not{q} \gamma^\nu \not{q}}{(k^2 - M^2)^2}. \quad (46)$$

Evaluating the traces for D -dimensional Dirac matrices and a d -dimensional integration over k gives

$$\begin{aligned} \text{tr}(\gamma^\mu \not{k} \gamma^\nu \not{k}) &= \text{tr}(-\gamma^\mu \gamma^\nu k^2 + 2\gamma^\mu \not{k} k^\nu) \\ &= \text{tr} \left(-\gamma^\mu \gamma^\nu k^2 + \frac{2}{d} \gamma^\mu \gamma^\nu k^2 \right) \end{aligned}$$

$$= \left(\frac{2}{d} - 1\right) D\eta^{\mu\nu} k^2 \quad (47)$$

$$\text{tr}(\gamma^\mu \not{q} \gamma^\nu \not{q}) = -D\eta^{\mu\nu} q^2 + 2Dq^\mu q^\nu \quad (48)$$

The integrand with the k^2 from (47) may be transformed to resemble the rest of the integrand according to

$$\int_k \frac{k^2}{(k^2 - M^2)^2} \left(\frac{2}{d} - 1\right) = \int_k \frac{1}{k^2 - M^2} \left(\frac{2}{d} - 1\right) + \int_k \frac{M^2}{(k^2 - M^2)^2} \left(\frac{2}{d} - 1\right). \quad (49)$$

The first term of (49) may be Wick-rotated to Euclidean space and evaluated using integration by parts, to give

$$\begin{aligned} \int_k \frac{1}{k^2 - M^2} \left(\frac{2}{d} - 1\right) &= i \int \frac{d\Omega_d k^{d-1} dk}{-k^2 - M^2} \left(\frac{2-d}{d}\right) \\ &= \frac{i}{d} \int \frac{d\Omega_d dk k^2}{k^2 + M^2} (d-2) k^{d-3} \\ &= \frac{-i}{n} \int d\Omega_d dk k^{d-2} \frac{d}{dk} \frac{k^2}{k^2 + M^2} \\ &= -\frac{2}{d} \int_k \frac{M^2}{(k^2 + M^2)^2}. \end{aligned} \quad (50)$$

Adding this to the other term in (49) gives

$$\int_k \frac{k^2}{(k^2 - M^2)^2} \left(\frac{2}{d} - 1\right) = - \int_k \frac{M^2}{(k^2 - M^2)^2}. \quad (51)$$

Combining (51) with the integrals having numerator (48) gives the self-energy result, equation (28).

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